$$
\begin{aligned}
\partial_{q+1}\circ P_q(\sigma) &= \partial_{q+1}(S_{q+1}(\sigma\times 1_I))\Big(\sum_{i=0}^{q}(-1)^i[v_0,\cdots,v_i,v'_i,\cdots,v'_q]\Big)\\
&= S_{q+1}(\sigma\times 1_I)\Big(\partial_{q+1}\Big(\sum_{i=0}^{q}(-1)^i[v_0,\cdots,v_i,v'_i,\cdots,v'_q]\Big)\Big)\\
&= S_{q+1}(\sigma\times 1_I)\Big(\sum_{i=0^q}(-1)^i\partial_{q+1}[v_0,\cdots,v_i,v'_i,\cdots,v'_q]\Big)\\
&= S_{q+1}(\sigma\times 1_I)\Big(\sum_{i=1}^{q}(-1)^i\Big\{\sum_{j=0}^{i}(-1)^j[v_0,\cdots,v_{j-1},v_{j+1},\cdots v_i,v'_i,\cdots,v'_q]\\
&\qquad\qquad\qquad\qquad +\sum_{j=i}^{q}(-1)^{j+1}[v_0,\cdots,v_i,v'_i,\cdots,v'_{j-1},v'_{j+1},\cdots,v'_q]\Big\}\Big)\\
&= \sum_{i=0}^{q}\sum_{j=0}^{i}(-1)^{i+j}(\sigma\times 1_I)\circ[v_0,\cdots,v_{j-1},v_{j+1},\cdots v_i,v'_i,\cdots,v'_q]\\
&\quad +\sum_{i=0}^{q}\sum_{j=i}^{q}(-1)^{i+j+1}(\sigma\times 1_I)\circ[v_0,\cdots,v_i,v'_i,\cdots,v'_{j-1},v'_{j+1},\cdots,v'_q]\\
&= \sum_{i=0}^{q}\sum_{j=0}^{i-1}(-1)^{i+j}(\sigma\times 1_I)\circ[v_0,\cdots,v_{j-1},v_{j+1},\cdots v_i,v'_i,\cdots,v'_q]\\
&\quad +\sum_{i=0}^{q}\sum_{j=i+1}^{q}(-1)^{i+j+1}(\sigma\times 1_I)\circ[v_0,\cdots,v_i,v'_i,\cdots,v'_{j-1},v'_{j+1},\cdots,v'_q]\\
&\quad +\sum_{i=0}^{q}(\sigma\times 1_I)\circ[v_0,\cdots,v_{i-1},v'_i,\cdots,v'_q]-\sum_{i=0}^{q}(\sigma\times 1_I)\circ[v_0,\cdots,v_i,v'_{i+1},\cdots,v'_q]\\
&= \sum_{i=0}^{q}\sum_{j=0}^{i-1}(-1)^{i+j}(\sigma\times 1_I)\circ[v_0,\cdots,v_{j-1},v_{j+1},\cdots v_i,v'_i,\cdots,v'_q]\\
&\quad +\sum_{i=0}^{q}\sum_{j=i+1}^{q}(-1)^{i+j+1}(\sigma\times 1_I)\circ[v_0,\cdots,v_i,v'_i,\cdots,v'_{j-1},v'_{j+1},\cdots,v'_q]\\
&\quad +(\sigma\times 1_I)\circ[v'_0,\cdots,v'_q]-(\sigma\times 1_I)\circ[v_0,\cdots,v_q].\\
P_{q-1}\circ\partial_q(\sigma) &= P_{q-1}\Big(\sum_{j=0}^{q}(-1)^j\sigma\circ[v_0,\cdots,v_{j-1},v_{j+1},\cdots,v_q]\Big)\\
&= \sum_{j=0}^{q}(-1)^jP_{q-1}\big(\sigma\circ[v_0,\cdots,v_{j-1},v_{j+1},\cdots,v_q]\big)\\
&= \sum_{j=0}^{q}(-1)^jS_q\big((\sigma\circ[v_0,\cdots,v_{j-1},v_{j+1},\cdots,v_q])\times 1_I\big)\Big(\sum_{i=0}^{q-1}(-1)^i[v_0,\cdots,v_i,v'_i,\cdots,v'_{q-1}]\Big)\\
&= \sum_{j=0}^{q}\sum_{i=0}^{q-1}(-1)^{i+j}(-1)^{i+j}\big((\sigma\circ[v_0,\cdots,v_{j-1},v_{j+1},\cdots,v_q])\times 1_I\big)\circ[v_0,\cdots,v_i,v'_i,\cdots,v'_{q-1}] \qquad (*)
\end{aligned}
$$

ここで

$$
\begin{aligned}
&\big((\sigma\circ[v_0,\cdots,v_{j-1},v_{j+1},\cdots,v_q])\times 1_I\big)\circ[v_0,\cdots,v_i,v'_i,\cdots,v'_{q-1}]\\
&\qquad =\begin{cases}[v_0,\cdots,v_{j-1},v_{j+1},\cdots,v_{i+1},v'_{i+1},\cdots,v'_q] & (j\leqq i \text{ のとき})\\ [v_0,\cdots,v_i,v'_i,\cdots,v'_{j-1},v'_{j+1},\cdots,v'_q] & (j>i \text{ のとき})\end{cases}
\end{aligned}
$$

であるので

$$
\begin{aligned}
(*) &= \sum_{j=0}^{q}\sum_{i=j}^{q-1}(-1)^{i+j}(\sigma\times 1_I)[v_0,\cdots,v_{j-1},v_{j+1},\cdots,v_{i+1},v'_{i+1},\cdots,v'_q]\\
&\quad +\sum_{j=0}^{q}\sum_{i=0}^{j-1}(-1)^{i+j}(\sigma\times 1_I)[v_0,\cdots,v_i,v'_i,\cdots,v'_{j-1},v'_{j+1},\cdots,v'_q]\\
&= \sum_{j=0}^{q}\sum_{i=j+1}^{q}(-1)^{i+j+1}(\sigma\times 1_I)[v_0,\cdots,v_{j-1},v_{j+1},\cdots,v_i,v'_i,\cdots,v'_q]\\
&\quad +\sum_{j=0}^{q}\sum_{i=0}^{j-1}(-1)^{i+j}(\sigma\times 1_I)[v_0,\cdots,v_i,v'_i,\cdots,v'_{j-1},v'_{j+1},\cdots,v'_q]\\
&= -\sum_{i=0}^{q}\sum_{j=0}^{i-1}(-1)^{i+j}(\sigma\times 1_I)[v_0,\cdots,v_{j-1},v_{j+1},\cdots,v_i,v'_i,\cdots,v'_q]\\
&\quad +\sum_{i=0}^{q}\sum_{j=i+1}^{q}(-1)^{i+j}(\sigma\times 1_I)[v_0,\cdots,v_i,v'_i,\cdots,v'_{j-1},v'_{j+1},\cdots,v'_q]
\end{aligned}
$$